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Energy flow, partial equilibration, and effective temperatures in systems with slow dynamics

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We show that, in nonequilibrium systems with small heat flows, there is a time-scale-dependent effective temperature that plays the same role as the thermodynamical temperature in that it controls the direction of heat flows and acts as a criterion for thermalization. We simultaneously treat the case of stationary systems with weak stirring and of glassy systems that age after cooling and show that they exhibit very similar behavior provided that time dependences are expressed in terms of the correlations of the system. We substantiate our claims with examples taken from solvable models with nontrivial low-temperature dynamics, but argue that they have a much wider range of validity. We suggest experimental checks of these ideas.
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I. INTRODUCTION

No physical system is ever in thermodynamical equilibrium. When we apply thermodynamics or statistical mechanics, we idealize the situation by assuming that ''fast'' processes have taken place, and ''slow'' ones will not: hence, we define an observation time scale which distinguishes these two kinds of processes [1]. It follows that the same system can be at equilibrium on one scale, and out of equilibrium on another, and, more strikingly, that it can be at equilibrium, but exhibiting different properties, on two scales at once.

Since the assumption of thermal equilibrium lies at the heart of statistical mechanics, it is usually hard to make these considerations without a strong appeal to one's intuition. We show in the following that they can be made, in fact, quite precise for a class of systems characterized by very slow energy flows. These systems are out of equilibrium, either because they are very gently ''stirred,'' i.e., work is constantly done on them, or because they have undergone a quench from higher temperatures a long time ago.

The most typical example of such a system is a piece of glass that has been in a room at constant temperature for several months. Since the glass itself is not in equilibrium, we have, in principle, no right of talking about ''the temperature of the glass,'' but only about the temperature of the room. However, we may legitimately ask what temperature would indicate a thermometer brought into contact with the glass and, again, we would be very surprised if it did not coincide with the room temperature. We would be even more surprised if, putting two points of the glass in contact with both ends of a copper wire, a heat flow were established through it.

In other words, although we know that equilibrium thermodynamics does not apply for the glass, we implicitly assume that some concepts that apply for equilibrium are still relevant for it. This is not because the glass is ''near equilibrium'' but rather because it has been relaxing for a long time, and therefore thermal flows are small.

Many attempts have been done to extend the concepts of thermodynamics to nonequilibrium systems—such as systems exhibiting spatiotemporal chaos or weak turbulence [2,3]. In this context, Hohenberg and Shraiman [3] have defined an effective ''temperature'' for stationary nonequilibrium systems through an expression involving the response, the correlation, and the temperature of the bath. A closely related expression appears naturally in the theory of nonstationary systems exhibiting aging [4,5], such as glasses.

We show here that this expression indeed deserves the name of temperature, because (i) the effective temperature associated with a time scale is the one measured on the system by a thermometer, in contact with the system, whose reaction time is equal to the time scale, (ii) it determines the direction of heat flows within a time scale, and (iii) it acts as a criterion for thermalization.

We shall here consider simultaneously two different conditions in which a regime with small flows of energy exists.

(1) Ordinary thermodynamical systems in contact with a heat bath at temperature T that are slowly driven (''stirred'') mechanically. The driving force is proportional to a small number which we shall denote D. The observation time

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and the manner of the stirring are such that, for long enough times, the system enters a *stationary*, time-translational invariant (TTI) regime [6]: one-time average quantities are independent of time, two-time quantities depend only upon time differences, etc. Stationarity is a weaker condition than thermodynamical equilibrium, since it implies loss of memory of the initial condition but not all other properties that are linked with the Gibbs-Boltzmann distribution.

(2) Purely relaxational systems that have been prepared through some cooling procedure ending at time $t=0$ and are kept in contact with a heat bath at a constant temperature T up to a (long) waiting time t_w (as in the example of the glass). t_w is also usually called ‘‘annealing time’’ in the glass literature. In this case physical quantities need not be TTI, and in interesting cases they will keep a dependence upon t_w (and also, in many cases, upon the cooling procedure) for all later times $t=\tau+t_w$.

We shall treat in parallel the ‘‘weak stirring’’ ($D\to 0$) and the ‘‘old age’’ ($t_w\to\infty$) limits: both taken *after the thermodynamical limit of infinite number of degrees of freedom*. We show that they lead to the same behavior, from the point of view of thermalization and effective temperatures, provided that one expresses time dependences in terms of the correlations of the system [7].

As a test of our ideas, we discuss thermalization in the context of the mean-field theory of disordered systems, or the low-temperature generalization of the mode coupling equations, but the nature of our results makes us confident that they have a much wider range of validity.

In Sec. II, we recall the generalization of the fluctuation-dissipation relation to the nonequilibrium case. In Sec. III, we consider the reading of a thermometer coupled to a system: when the system is in equilibrium we show that the thermometer measures the temperature of the heat bath, while when it is out of equilibrium it measures different effective temperatures depending on the observation time scale. These effective temperatures are equal or higher than the one of the bath and are closely related to the FDT violation factor [8–13,4,5] introduced to describe the out-of-equilibrium dynamics of glassy systems. In Sec. IV, we recall how time scales or correlation scales are defined in systems with slow dynamics. We then argue that, if the FDT violation factor is well defined within a time scale, a single degree of freedom thermalizes within that time scale to the corresponding effective temperature. In Sec. V, we extend this analysis to several degrees of freedom and show that the effective temperature determines the direction of heat flows, and can be used as a thermalization criterion. In Sec. VI, we discuss various phenomenological ‘‘fictive temperature’’ ideas that have been used for a long time in the theory of structural glasses. Our conclusions are summarized in Sec. VII, where some experimental implications of our work are suggested.

II. THE FLUCTUATION-DISSIPATION RELATION OUT OF EQUILIBRIUM

Let us consider a system with N degrees of freedom $(s_1,\ldots,s_N)$, whose dynamics is described by Langevin equations of the form

$$\dot{s}_i=b_i(s)+\eta_i(t), \tag{2.1}$$

where $\eta_i(t)$ is the Gaussian thermal noise. For *unstirred* systems, we consider purely relaxational dynamics, where the average velocity $b_i(s)$ is proportional to the gradient of the Hamiltonian $E(s)$:

$$b_i(s)=-\sum_j \Gamma_{ij}\frac{\partial E(s)}{\partial s_j}. \tag{2.2}$$

The symmetric matrix Γ is related to the correlation function of the noise η by the Einstein relations

$$\langle \eta_i(t)\eta_j(t')\rangle=2T\Gamma_{ij}\delta(t-t'), \tag{2.3}$$

where T is the temperature of the heat bath. Averages over the thermal history, i.e., averages over many realizations of the same experiment with different realizations of the heat bath, will be denoted by angular brackets. We assume of course $\langle \eta_i(t)\rangle=0$, $\forall i,t$. We have chosen the temperature units so that Boltzmann’s constant is equal to 1. The equilibrium distribution is then proportional to the Boltzmann factor $\exp(-E/T)$.

For *stirred* systems, we add to b_i a perturbation proportional to D, that cannot be represented as the gradient of a function (i.e., is not purely relaxational), but is otherwise generic. We then have $W\equiv\langle\Sigma_i b_i(s)s_i\rangle>0$ at stationarity, meaning that work is being done on the system [14,15].

We denote the observables (energy, density, magnetization, etc.) by $O(s)$. Throughout this work we shall denote by t_w or t' the earliest time (to be related to the waiting time), t the latest time, and τ the relative time $t-t_w$. These times are measured, in the case of the unstirred systems which exhibit aging, from the end of the cooling procedure.

Given two observables O_1 and O_2, we define their correlations $C_{12}(t,t_w)\equiv\langle O_1(t)O_2(t_w)\rangle-\langle O_1(t)\rangle\langle O_2(t_w)\rangle$, and their mutual response

$$R_{12}(t,t_w)\equiv\frac{\delta O_1(t)}{\delta h_2(t_w)}, \tag{2.4}$$

where h_2 appears in a perturbation of the Hamiltonian of the form $E\to E-h_2(t)O_2$. Obviously, causality implies $R_{12}(t,t_w)=0$ for $t<t_w$. It is also useful to introduce the integrated response (susceptibility)

$$\chi_{12}(t,t_w)\equiv\int_{t_w}^{t}dt' R_{12}(t,t'). \tag{2.5}$$

Let us now make a parametric plot [5] of $\chi(t,t_w)$ vs $C(t,t_w)$ for several increasing values of t_w. We thus obtain a limit curve $\lim_{t_w\to\infty}(t,t_w)=\chi(C)$. In the case of a weakly driven system, we wait for stationarity and plot $\chi(t-t_w)$ vs $C(t-t_w)$ for several decreasing values of the driving D. We thus obtain the curve $\lim_{D\to 0}\chi(t-t_w,D)=\chi(C)$.

The fluctuation-dissipation theorem (FDT) relates the response and correlation function at equilibrium. One has

$$R_{12}(t-t_w)=\frac{1}{T}\frac{\partial C_{12}(t-t_w)}{\partial t_w}, \tag{2.6a}$$

$$\chi_{12}(t-t_w)=\frac{1}{T}[C_{12}(0)-C_{12}(t-t_w)]. \tag{2.6b}$$

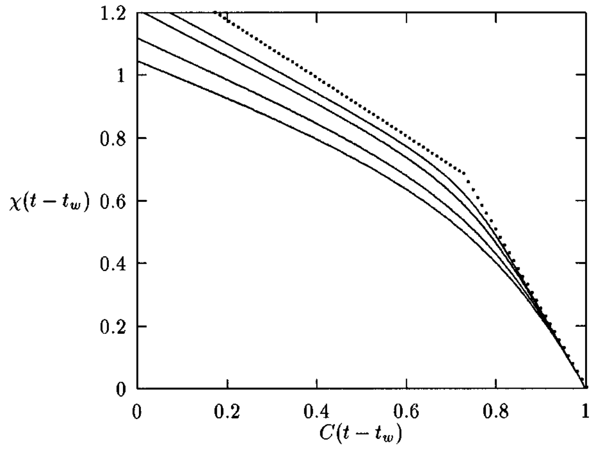

FIG. 1. The susceptibility $\chi(t-t_w)$ vs the autocorrelation function $C(t-t_w)$ for the model of Appendix A once stationarity is achieved. The parameter D is equal to 0.05, 0.375, 0.025, and 0.0125, respectively, from bottom to top. The dots represent the analytical solution for the limit $D \to 0$. One sees that, in this limit, the FDT violation factor $X(C)$ tends *continuously* to the dotted straight lines. The value of C at the breakpoint is C^{EA}, the Edwards-Anderson order parameter or the ergodicity breaking parameter in the language of the MCT.

If the equilibrium distribution is asymptotically reached for $t_w \to \infty$ (or $D \to 0$ in the case of stirred systems) the FDT implies that the limit curve $\chi(C)$ is a straight line of slope $-1/T$.

However, there is a family of systems for which the limiting curve $\chi(C)$ *does not* approach a straight line (see Figs. 1 and 2). For driven systems this means that the slightest stirring is sufficient to produce a large departure from equilibrium *even at stationarity* (Fig. 1), while in the case of a relaxational system it means that the system is unable to equilibrate within experimental times (Fig. 2).

Let us denote by $-X(C)/T$ the slope of the curve $\chi(C)$:

$$\frac{d\chi(C)}{dC} \equiv -\frac{X(C)}{T}. \tag{2.7}$$

This corresponds to

$$R(t,t_w) = \frac{X(C)}{T}\frac{\partial C(t,t_w)}{\partial t_w}, \tag{2.8}$$

where the derivative is taken with respect to the earlier time. We have thus defined $X(C)$, the FDT violation factor [4,5], for nonequilibrium systems with slow dynamics.

When FDT holds, for $D \to 0$ (or $t \geqslant t_w \to \infty$, respectively) we can treat the system as being in equilibrium and X tends to 1 in the limit. When this does not happen, we may inquire about the physical meaning of $X(C)$. In order to answer this question, let us first recall the relationship there is between the FDT and the equipartition of energy.

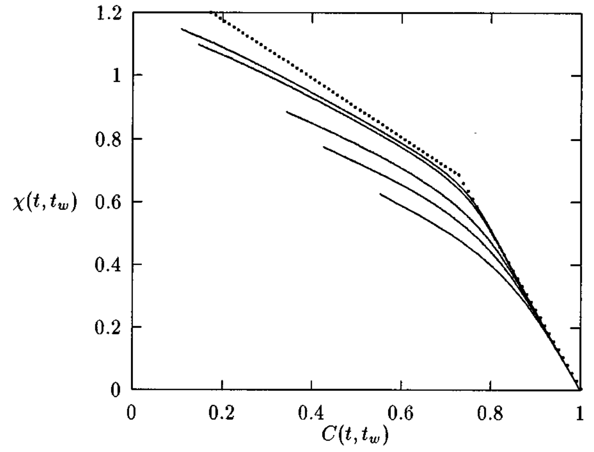

FIG. 2. The susceptibility $\chi(t,t_w)$ vs the autocorrelation function $C(t,t_w)$ for the model of Appendix A at $T<T_g$ (D is strictly zero). The full curves correspond to different total times t, equal, from bottom to top, to 12.5, 25, 37.5, 50, and 75, respectively ($t_w > t/4$ throughout). The dots represent the analytical solution when $t_w \to \infty$. Neither $\chi(t,t_w)$ nor $C(t,t_w)$ achieve stationarity (see Fig. 3).

III. FREQUENCY-DEPENDENT THERMOMETERS THAT MEASURE EFFECTIVE TEMPERATURES

We use a harmonic oscillator of frequency ω_0 to measure the ‘‘temperature’’ of a degree of freedom $O(s)$ [$O(s)$ may be the energy, or some spatial Fourier component of the magnetization]. At the waiting time t_w we weakly couple the oscillator to the system via $O(s)$, while we keep the system in contact with a heat bath at temperature T. We wait for a short time until the average energy of the oscillator has stabilized. If the system were in equilibrium, by the principle of equipartition of energy we would have $\langle E_{\mathrm{osc}} \rangle = T$.

Assuming linear coupling, the Hamiltonian reads

$$E_{\mathrm{total}} = E(s) + E_{\mathrm{osc}} + E_{\mathrm{int}}, \tag{3.1}$$

where

$$E_{\mathrm{osc}} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2, \tag{3.2}$$

$$E_{\mathrm{int}} = -aO(s)x. \tag{3.3}$$

The equation of motion of the oscillator reads

$$\ddot{x} = -\omega_0^2 x + aO(t). \tag{3.4}$$

In the presence of the coupling, if $ax(t)$ is sufficiently small (an assumption we have to verify *a posteriori*), we can use linear response theory to calculate the action on O of the oscillator:

$$O(t) = O_b(t) = a\int_0^t dt' R_O(t,t')x(t'), \tag{3.5}$$

where $O_b(t)$ is the fluctuating term and where the response function R_O is defined by

$$R_O(t,t') = \frac{\delta\langle O\rangle(t)}{\delta a x(t')}. \tag{3.6}$$

We assume moreover that the average $\langle O_b(t)\rangle$ exists (we set it to zero by a suitable shift of x) and that the fluctuations of O_b (in the absence of coupling) are correlated as

$$\langle O_b(t)O_b(t_w)\rangle = C_O(t,t_w), \tag{3.7}$$

where $C_O(t,t_w)$ is a quantity of $O(N)$.

The equation for x then reads

$$\ddot{x} = -\omega_0^2 x + aO_b(t) + a^2\int_0^t dt' R_O(t,t')x(t'). \tag{3.8}$$

Thus the oscillator takes up energy from the fluctuations of O, and dissipates it through the response of the system. Equation (3.8) is linear and easy to solve in the limit of small a^2 by Fourier-Laplace transform. One thus obtains the following results, whose proof is sketched in Appendix B.

Consider first the case of stirred systems at stationarity, in which both the correlation and the response are TTI. The average potential energy of the oscillator reaches the limit

$$\frac{1}{2}\,\omega_0^2\langle x^2\rangle = \frac{1}{2}\,\langle E_{\text{osc}}\rangle = \frac{\omega_0 C_O(\omega_0)}{2\chi_O''(\omega_0)}, \tag{3.9}$$

after a time $\sim t_c$ given by

$$t_c = \frac{2\omega_0}{a^2\chi''(\omega_0)}. \tag{3.10}$$

We have defined

$$\chi''(\omega) \equiv \text{Im}\int_0^\infty dt\; R(t)e^{i\omega t},$$

$$\tilde{C}(\omega) \equiv \text{Re}\int_0^\infty dt\; C(t)e^{i\omega t}. \tag{3.11}$$

If we now define the temperature $T_O(\omega)$ as

$$T_O(\omega) \equiv \langle E_{\text{osc}}\rangle, \tag{3.12}$$

we obtain

$$T_O(\omega) = \frac{\omega_0\tilde{C}_O(\omega_0)}{\chi_O''(\omega_0)}. \tag{3.13}$$

This is precisely the temperature defined by Hohenberg and Shraiman [3] for the case of weak turbulence [the spatial dependence is encoded in $O(s)$].

Let us now turn to the case of relaxational dynamics where TTI is violated. Here we have to take into account the time t_w at which the measurement is performed, and consider $\omega_0^{-1} \ll t_c \ll t_w$. Then, a similar calculation yields for the energy of the oscillator

$$\frac{1}{2}\,\omega_0^2\langle x^2\rangle_{t_w} = \frac{1}{2}\,\langle E_{\text{osc}}\rangle_{t_w} = \frac{\omega_0\tilde{C}_O(\omega_0,t_w)}{2\chi_O''(\omega_0,t_w)}, \tag{3.14}$$

where the average $\langle x^2\rangle_{t_w}$ is taken on a comparatively short time stretch after t_w. This definition of the frequency- and waiting-time-dependent correlation $C_O(\omega_0,t_w)$ and out-of-phase susceptibility $\chi_0''(\omega_0,t_w)$ closely follows the actual experimental procedure for their measure: one considers a time window around t_w consisting of a few cycles (so that phase and amplitude can be defined) and small enough respect to t_w so that the measure is ''as local as possible in time.'' In fact, these two quantities are standard in the experimental investigation of aging phenomena in spin glasses [16,18].

The natural definition of the frequency- and time-dependent temperature of O is then

$$T_O(\omega_0,t_w) \equiv \frac{\omega_0\tilde{C}_O(\omega_0,t_w)}{\chi_O''(\omega_0,t_w)}. \tag{3.15}$$

If equilibrium is achieved, then the temperature is independent of t_w, of the frequency ω_0 and of the observable O, and coincides with that of the heat bath. The index O recalls that the effective temperature may depend on the observable. The frequency-dependent temperature defined by either Eq. (3.13) or (3.15) is compatible with the Fourier transformed expression of the FDT violation factor (2.8) provided that it does not vary too fast with ω_0. We show later that this is indeed the case for systems with slow dynamics.

For this definition we have chosen somehow arbitrarily an oscillator as our thermometer. However, we show in Appendix C that the same role can be played by any small but macroscopic thermometer, weakly coupled to the system. The role of the characteristic frequency ω_0 is then played by the inverse of the typical response time of the thermometer.

Now $T_O(\omega,t_w)$ only deserves to be called a ''temperature'' if it controls the direction of heat flow. A first way to check whether this is the case is to consider an experiment in which we connect the oscillator to an observable O_1, and let it equilibrate at the temperature $T_{O_1}(\omega,t_w)$. We then disconnect it and connect it to another observable O_2 and let it equilibrate at the temperature $T_{O_2}(\omega,t_w)$. This is not like a Maxwell demon, since the times of connection and disconnection are unrelated to the microscopic behavior of the system. The net result is that an amount of energy $T_{O_1}(\omega,t_w) - T_{O_2}(\omega,t_w)$ has been transferred from the degrees of freedom associated with O_1 to those associated with O_2: therefore the flow goes from high to low temperatures. This is an actual realization of the idea of ''touching two points of the glass with a copper wire'' described in the Introduction.

This observation also suggests a possible explanation of the fact that all FDT violation factors that we know of are smaller than one: if $T_O(\omega,t_w)$ were smaller than the temperature of the heat bath, it could be possible, in principle, to extract energy from the bath by connecting between it and our system a small Carnot engine. The argument can be made sharper by considering a stationary situation in a weakly stirred system: but to argue that this situation would lead to a violation of the second principle one needs to prove that the equilibration times of the Carnot engine are short enough to make the power it produces larger than the dissipated one.

IV. EQUILIBRATION WITHIN A TIME SCALE

Before discussing the effective temperature as an equilibration factor we need to introduce some general features of the time evolution of systems with slow dynamics. We first define the time correlation scales and we then argue that, if the FDT violation factor is well defined within a time scale, a single degree of freedom thermalizes within that time scale at the corresponding effective temperature.

A. Correlation and response scales

Systems having a long-time out-of-equilibrium dynamics tend to have different behaviors in different time scales. Let us start by describing them for long-time dynamics in the purely relaxational case. In Appendix D we give a formal definition of correlation scale, following [5].

Consider first, as an example, the dynamics of domain growth [19] for the Ising model at low but nonzero temperatures. The autocorrelation function $C(t,t_w) = (1/N)\Sigma_i s_i(t) s_i(t_w)$ exhibits two regimes.

At long times t,t_w, such that $t-t_w \ll t_w$, the correlation function shows a fast decay from 1 (at equal times) to m^2, where m is the magnetization. This regime describes the fast relaxation of the spins within the bulk of each domain.

At long and well-separated times $(t-t_w \sim t_w)$, the correlation behaves as a function of $L(t_w)/L(t)$, where $L(t)$ is some measure of the typical domain size at time t.

We refer to these two scales as ''quasiequilibrium'' and ''coarsening'' (or ''aging''), respectively. After a given time, the correlation rapidly decays to a plateau value m^2, and the speed with which it falls below that value becomes smaller and smaller as t_w grows.

This example helps to stress the fact that different scales are defined as a function of both times [in this case $t-t_w$ finite, $L(t_w)/L(t)$ finite] and are well separated only in the limit where both times are large.

In Fourier space, this separation of scales is achieved by considering several frequencies ω and increasing times t_w. Then, if we consider $\omega \sim$const for increasing t_w, we probe the quasiequilibrium scale, while if we want to probe the aging or coarsening scale we have to consider smaller and smaller ω, keeping $\omega L(t_w)/L'(t_w) \sim$const.

If we now consider two frequencies ω_1 and ω_2 and keep

$$\frac{\omega_1}{\omega_2} \sim \text{const}, \tag{4.1}$$

we shall probe, as $t_w \to \infty$, the *same* scale: it will be the quasiequilibrium scale if they both remain finite, or the coarsening (aging) one if we keep $\omega_{1,2} L(t_w)/L'(t_w) \sim$const.

These considerations can be generalized to other systems with slow dynamics. In general there may be more than two relevant scales [9,4,5,20], for example ω=const, $\omega t_w^{1/2}$=const, ωt_w=const, etc. In any case, the condition for looking into the same scale via two successions of frequencies remains Eq. (4.1).

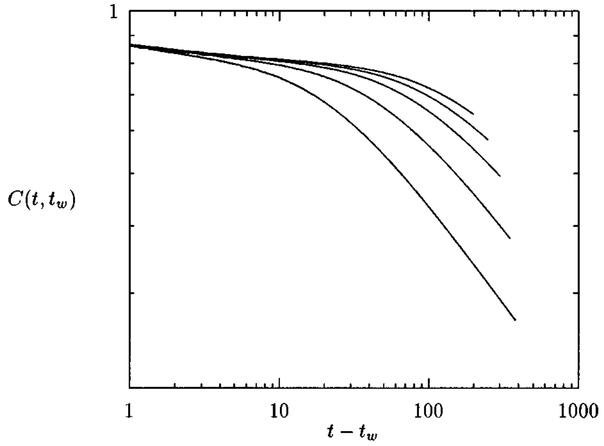

FIG. 3. The correlation function $C(t,t_w)$ vs $t-t_w$ for the same model as in Fig. 2. From bottom to top t_w=20, 50, 100, 150, and 200. The correlation decays rapidly from 1 to $C^{\rm EA} \sim 0.73$ and then more slowly from $C^{\rm EA}$ to 0. This second decay becomes slower and slower as t_w increases.

In Fig. 3 we show the numerical solution of our test model, defined in Appendix A. The autocorrelation function $C(t,t_w)$ is plotted vs the time difference $t-t_w$ for several waiting times in log-log scale. These plots (which are standard in the Monte Carlo simulations of spin glasses [21]), show (i) that the system is out of equilibrium, since we have an explicit dependence on t_w, and (ii) that there are at least two time correlation scales. For short time differences $t-t_w$ the decay is fast, the autocorrelation function is TTI and it falls from 1 to $C^{\rm EA}$. It then decays further from $C^{\rm EA}$ to zero, more and more slowly as the waiting time increases. The quantity $C^{\rm EA}$ is known in the language of spin glasses as the Edwards-Anderson order parameter and in mode coupling theory (MCT) as the nonergodicity parameter: it measures the strength of the fast correlations ($C^{\rm EA}=m^2$ in domain growth). Quasiequilibrium and aging regimes are experimentally observed in real spin glasses and polymer glasses [22,18].

Let us now turn to a similar analysis for driven systems in the limit of weak driving energy $D \to 0$. In that case, even if the system reaches a TTI regime provided we wait long enough (the amount of waiting increases when the stirring rate D decreases), it sometimes happens that some correlations and responses acquire a nontrivial low-frequency behavior in the limit $D \to 0$ [23]. For example, in Fig. 4 we show the same plot as in Fig. 3, i.e., $C(t-t_w)$ vs $t-t_w$, for different, but small, stirring rates. We see that there are at least two time scales: one, for short time differences, where the correlation decays rapidly from 1 to $C^{\rm EA}$ and one, for long time differences, in which it slowly decays from $C^{\rm EA}$ to

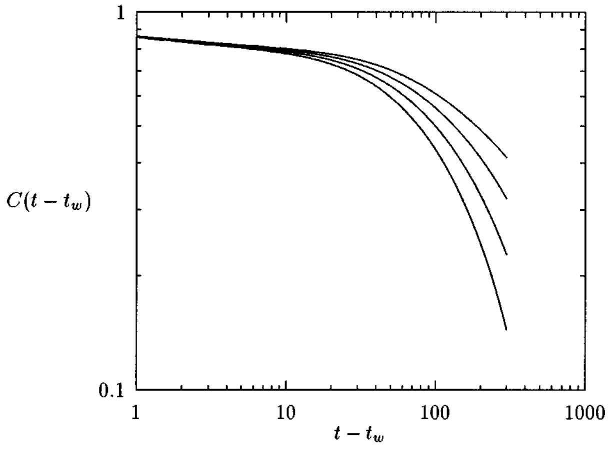

FIG. 4. The correlation function $C(t-t_w)$ vs $t-t_w$ for the same model as in Fig. 1. From bottom to top D=0.1, 0.075, 0.05, and 0.025. The correlation decays rapidly from 1 to $C^{\rm EA} \sim 0.73$ and then more slowly from $C^{\rm EA}$ to 0. This second decay becomes slower and slower as D decreases.

zero. The smaller the stirring, the slower the decay of C from C^{EA} to zero.

It is useful to consider, in these cases, frequencies that go to zero as some function of D. The condition that two sequences of frequencies ω_1,ω_2 correspond to the same scale now reads

$$\lim_{D\to 0}\frac{\omega_1}{\omega_2}=\text{const.} \tag{4.2}$$

B. Thermalization criterion for a single degree of freedom within a scale

Let us now consider a system at a given waiting time t_w or stirring rate D. Heat flows tend to zero as $t_w\to\infty$ or $D\to 0$, respectively. It is reasonable to assume that, in these limits, the *effective temperatures* associated with any given observable O on a given time scale tend to equalize, provided that their limit is finite. We thus have, for example, in the case of relaxational dynamics, for $\omega_1/\omega_2=$const,

$$\lim_{\substack{t_w\to\infty\\ \omega_1\to 0}} T_O(\omega_1,t_w)=\lim_{\substack{t_w\to\infty\\ \omega_2\to 0}} T_O(\omega_2,t_w). \tag{4.3}$$

Similarly, we expect that each fixed frequency sooner or later thermalizes with the bath:

$$\lim_{\substack{t_w\to\infty\\ \omega=\text{const}}} T_O(\omega,t_w)=T. \tag{4.4}$$

In some important cases the effective temperatures defined by Eqs. (3.13) and (3.15) diverge in the limit $t_w\to\infty$ ($D\to 0$). In these cases the effective temperature $T_O(\omega,t_w)$ should diverge for the whole time scale. We shall dwell on this problem in Sec. IV C.

Equations (4.3) and (4.4) are trivially true in a system that reaches thermal equilibrium, where all effective temperatures eventually reach the temperature of the reservoir. However, they also describe the situation in which smaller frequencies take longer to reach the temperature of the heat bath, in such a way that at any given (long) waiting time there are low enough frequencies that have a temperature substantially different (in all cases we know, higher) from that of the bath. In particular, Eq. (4.4) allows us to answer a question we asked at the beginning: if a piece of glass has been kept at room temperature for several months, a thermometer whose response time is of order of a few seconds would measure the room temperature, but it would read a higher temperature if its response time is of the order of weeks.

In fact, with the appropriate handling of time scales [5] (see Appendix D), Eqs. (4.3) and (4.4) make it possible to calculate the out-of-equilibrium relaxation of mean-field spin glasses, and also the low-temperature generalization of the mode-coupling equations for one single mode. We thus obtain a solvable example where Eqs. (4.3) and (4.4) hold.

For the case of stirred systems, the $D\to 0$ limit plays the same role as the $t_w\to\infty$ limit in relaxational systems provided that the time scales are suitably redefined as in Eq. (4.2). In particular, at each fixed value of ω one has $\lim_{D\to 0}T_O(\omega,D)=T$, and, for $\omega_1/\omega_2=$const one has

$$\lim_{\substack{D\to 0\\ \omega_1\to 0}} T_O(\omega_1,D)=\lim_{\substack{D\to 0\\ \omega_2\to 0}} T_O(\omega_2,D), \tag{4.5}$$

provided that both limits are finite.

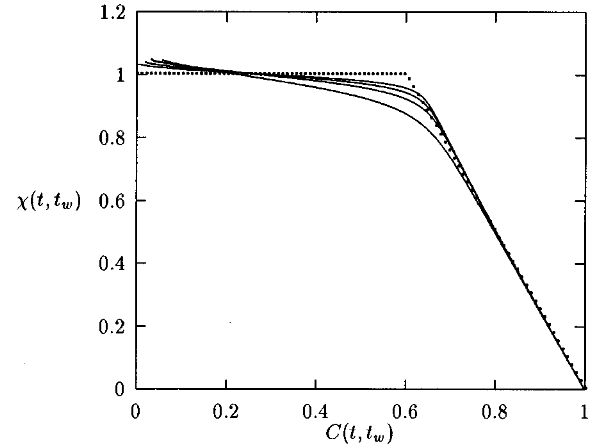

FIG. 5. The susceptibility $\chi(t,t_w)$ vs the autocorrelation function $C(t,t_w)$ for the purely relaxational $p=2$ spherical model [equivalent to the $O(N)$, $N\to\infty$ ferromagnetic coarsening in $d=3$]. The dots represent the analytical solution when $t_w\to\infty$. The total time t is equal to 20, 50, 100, and 200 ($t_w>t/4$ throughout).

If one looks into a time scale within which the temperature is almost constant, one can indifferently use a temperature defined in terms of a frequency or a time. In the relaxational case one has

$$T_O(\omega,t_w)=T_O(t,t_w), \tag{4.6}$$

provided that $t-t_w\sim\omega^{-1}$. In the same situation one has

$$R(t,t_w)=\frac{1}{T_O(t,t_w)}\frac{\partial C(t,t_w)}{\partial T_w}. \tag{4.7}$$

Similar relations hold for the driven case. If, on the contrary, one considers values of frequency or time for which the temperature defined by Eq. (4.7) is not constant, one cannot directly relate it with the reading of a thermometer coupled to the system.

C. Coarsening and the case of infinite effective temperature

An important physical situation which deserves a special discussion is that of domain growth. In this case the effective temperature (associated, say, with the total magnetization) in the "coarsening scale," $\omega L(t_w)/L'(t_w)=$const, tends to infinity as $t_w\to\infty$. The curve χ (magnetic susceptibility) C (magnetization correlation) looks like in Fig. 5. This figure shows the results for a model that is equivalent to the $O(N)$ ferromagnetic coarsening in three dimensions. The integrated response χ becomes flat, as $t_w\to\infty$, in the aging regime. In other words, the long-term memory *tends to disappear*.

Because experimental measures of aging are in general related to the response, this kind of system is sometimes referred to as exhibiting aging "in the correlations" (correlations are not TTI) but not in the response. One should stress, however, that the divergence of the effective temperature in the aging time scale can be extremely slow: for example, in the Fisher-Huse model for spin glasses [24]

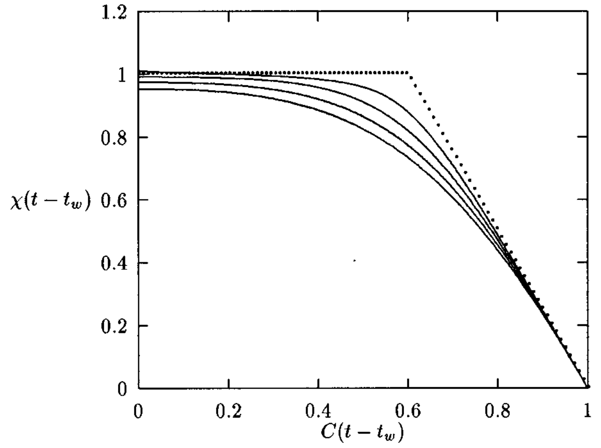

FIG. 6. The susceptibility $\chi(t-t_w)$ vs the autocorrelation function $C(t-t_w)$ for the $p=2$ spherical asymmetric model [''stirred'' $O(N)$, $N\to\infty$ ferromagnetic coarsening in $d=3$] for different levels of asymmetry (stirring): from bottom to top $D=0.8$, 0.6, 0.4, and 0.2. The dots represent the analytical solution in the limit of zero asymmetry.

$T_O(\omega,t_w)$ grows like a power of $\ln t_w$.

The fact that the effective temperature in the aging regime tends to infinity means that if we measure it at ω small and fixed, the temperature $T(\omega,t_w)$ grows with t_w (while we are still probing the aging regime) and then starts falling, finally reaching the temperature of the bath. If we repeat the experiment with a smaller ω, the overall behavior is the same, but the highest temperature reached will be higher; and so on without an upper limit. The situation is similar, but experimentally more subtle, for ''stirred'' systems for which the effective temperatures diverge as $D\to 0$. In such cases we measure a system that has a given (small) driving D. If the system is such that for $D\to 0$ the effective temperature of all but the fastest scale tend to infinity, we may never realize that this is the case in an experimental situation, as long as we are not able to repeat the experiment with smaller and smaller stirring.

We show one such case in Fig. 6, which represents the same coarsening problem as before, but made stationary by a ''stirring'' term in the equation of motion proportional to D. If one had performed an experiment on such a system for a fixed D, one would have observed high effective temperatures for separated times, and no evidence of thermalization. Only by letting D take smaller and smaller values one can notice that in fact these temperatures tend to infinity.

D. Systems with finite effective temperatures

In mean-field models one can close the Schwinger-Dyson equations into a set of dynamical equations involving only the correlation and response functions. This allows us to solve them analytically for large times, and also to obtain their full solution, via a numerical integration, starting from a pair of initial times $t_w=t=0$. We thus obtain the curve $\chi(C)$ by integrating the response function and plotting it vs C. Two cases in which there are only two time-correlation scales are the test model [4] considered here (cf. Appendix A) and the case of a particle moving in an infinite-dimensional random medium with short-range correlations [25]. Both the analytical and the numerical results exhibit thermalization within the aging regime.

In the Sherrington-Kirkpatrick spin-glass model there is a full hierarchy of time scales and effective temperatures, a fact also confirmed numerically [26] and by Monte Carlo simulations [5]. Another model with infinite many scales and with a full hierarchy of effective temperatures is that of a particle moving in an infinite-dimensional random medium with long-range correlations [20,25].

In Ref. [27], the Monte Carlo simulation of the ''realistic'' 3D Edwards-Anderson model for a spin glass was used to obtain the $\chi(C)$ curves, which seem to approach a nontrivial curve for increasing t_w. This suggests the existence of a hierarchy of time-correlation scales.

The ''trap model'' for spin-glass dynamics [28] violates Eq. (4.3) when an unusual choice of a parameter (α) is made. However, it is difficult to interpret the model, with this choice of parameter, as a phenomenological model stemming from a reasonable microscopic dynamics.

V. THERMALIZATION OF DIFFERENT DEGREES OF FREEDOM WITHIN A TIME SCALE

As we remarked in Sec. III, a *bona fide* temperature should control heat flow and thermalization. It is the aim of this section to show that this is indeed the case for the effective temperature we have defined. If this were not the case, it would be possible to use our small oscillator to transfer heat from some degrees of freedom to others: in other words, by decorating our copper wire with a suitable frequency filter, we would observe heat flowing through it when it touches the two ends of the glass.

We shall argue that if different degrees of freedom *effectively interact* on a given time scale, then they thermalize on that scale. A useful—though not universal—criterion for ''effective interaction'' is that their mutual response (the response of one of the degrees of freedom to an oscillating field of the given frequency conjugate to the other degree of freedom) is of the same order as their self-response on that time scale.

The argument is essentially the same both for relaxational systems that have evolved for a long time, or in stationary driven systems in the limit of small driving energy. We shall thus focus on the relaxational case only.

We emphasize again that the thermalization of different degrees of freedom is well defined only in the limit of vanishing heat flow, i.e., long waiting times or vanishing stirring rates, respectively. This is witnessed by the appearance of one (or more) well-separated plateaus in the decay of the two-time correlation functions.

Let us consider n modes (labeled by $a,b=1,\dots,n$). One can write, in general, some Schwinger-Dyson equations for their correlations $\mathsf{C}=(C_{ab})$ and responses $\mathsf{R}=(R_{ab})$:

$$\begin{aligned}\frac{\partial C_{ab}(t,t_w)}{\partial t}=&-\sum_c \mu_{ac}(t)C_{cb}(t,t_w)+2TR_{ab}(t_w,t)\\&+\sum_c\int_0^{t_w}dt''D_{ac}(t,t'')R_{cb}(t_w,t'')\\&+\sum_c\int_0^{t}dt''\Sigma_{ac}(t,t'')C_{cb}(t'',t_w),\qquad(5.1a)\end{aligned}$$

$$\frac{\partial R_{ab}(t,t_w)}{\partial t} = -\sum_c \mu_{ac}(t) R_{cb}(t,t_w) + \delta(t-t_w)\delta_{ab} + \sum_c \int_{t_w}^{t} dt'' \Sigma_{ac}(t,t'') R_{cb}(t'',t_w). \tag{5.1b}$$

As they stand, Eqs. (5.1) are just a way to hide our difficulties under the Σ, D carpet. Several approximations extensively used in the literature amount to various approximations of the kernels Σ and D.

In equilibrium, the symmetries [29] of the original problem allow us to write

$$\Sigma_{ab}(t-t_w) = \frac{1}{T}\frac{\partial D_{ab}}{\partial t_w}(t-t_w), \tag{5.2a}$$

$$R_{ab}(t-t_w) = \frac{1}{T}\frac{\partial C_{ab}}{\partial t_w}(t-t_w). \tag{5.2b}$$

If we now make for Σ and D the approximation that are ordinary functions (instead of general functionals) of the correlations and the responses, we obtain the mode-coupling approximation (MCA)

$$D_{ab}(\mathsf{C}) = F_{ab}(\mathsf{C}), \quad \Sigma_{ab}(\mathsf{C}) = \sum_{c,d} F_{ab,cd}(\mathsf{C}) R_{cd}, \tag{5.3}$$

where there is a model-dependent function $F(\mathsf{q})$ such that

$$F_{ab}(\mathsf{q}) = \frac{\partial F}{\partial q_{ab}}, \quad F_{ab,cd}(\mathsf{q}) = \frac{\partial^2 F}{\partial q_{ab}\partial q_{cd}}. \tag{5.4}$$

If the system equilibrates, one recovers the usual form of the mode-coupling theory (MCT) that is applied for supercooled liquids [30]. It will be instructive to recall how equilibrium is reached within the framework of Eq. (5.4). One recalls the FDT conditions (5.2); time translational invariance: functions of one time are just constant and functions of two times depend upon time differences only; reciprocity: $C_{ab}(t-t_w) = C_{ba}(t-t_w)$. Putting this information in Eqs. (5.1) and (5.4) one obtains

$$\frac{\partial C_{ab}(t-t_w)}{\partial t} = -\sum_c \mu_{ac} C_{cb}(t-t_w) + \frac{1}{T}\sum_c [D_{ac}(0) C_{cb}(t_w-t_w) - D_{ac}^{\infty} C_{cb}^{\infty}] + \frac{1}{T}\sum_c \int_{t_w}^{t} dt'' D_{ac}(t-t'') \frac{\partial C_{cb}(t''-t_w)}{\partial t''}, \tag{5.5}$$

where $D_{ac}^{\infty}, C_{cb}^{\infty}$ stand for limits of widely separated times, i.e.,

$$C_{\mathrm{ac}}^{\infty} \equiv \lim_{t\to\infty} C_{\mathrm{ac}}(t), \quad D_{\mathrm{ac}}^{\infty} \equiv \lim_{t\to\infty} D_{\mathrm{ac}}(t). \tag{5.6}$$

This is the usual equilibrium MCT equation [30]. The equation for the response (5.1b) with the same equilibrium assumptions becomes the time derivative (divided by T) of Eq. (5.1a), as it should.

In a case in which the system is unable to reach thermal equilibrium, like the low-temperature phase of spin glasses, the solution of Eqs. (5.1) will exhibit several time scales. The question as to how many scales one has to consider in order to close the dynamical equations can be answered for each model unambiguously by the construction in Ref. [5], suitably generalized to several modes.

Here, for simplicity, we assume that there are only two relevant time scales (as in the coarsening example of the last section). We discuss later a model system that acts as an explicit example of this situation. We propose an ansatz for the long-time asymptotics, and then verify that it closes the equations [4]. We assume (and later verify) that all two-time functions can be separated in two regimes.

Finite time differences with respect to the long waiting time, i.e., $t-t_w$ finite and positive, and $t_w \to \infty$. In this regime TTI and FDT hold.

Aging regime, corresponding to long and widely separated times, i.e., $t \sim t_w \to \infty$. In this regime neither TTI nor FDT hold.

For finite time differences and a long waiting time t_w we have

$$C_{ab}^{\mathrm{FDT}}(t-t_w) \equiv \lim_{t_w\to\infty} C_{ab}(t,t_w), \tag{5.7a}$$

$$R_{ab}^{\mathrm{FDT}}(t-t_w) \equiv \lim_{t_w\to\infty} R_{ab}(t,t_w), \tag{5.7b}$$

with

$$R_{ab}^{\mathrm{FDT}}(t-t_w) = \frac{1}{T}\frac{\partial C_{ab}^{\mathrm{FDT}}}{\partial t_w}(t-t_w), \tag{5.8}$$

$$C_{ab}^{\mathrm{EA}} \equiv \lim_{t-t_w\to\infty} \lim_{t_w\to\infty} C_{ab}^{\mathrm{FDT}}(t-t_w). \tag{5.9}$$

The aging regime is defined as the time domain in which the correlations fall (more and more slowly) below C_{ab}^{EA}. For these times we denote

$$C_{ab}(t,t_w) = \tilde{C}_{ab}(t,t_w), \quad R_{ab}(t,t_w) = \tilde{R}_{ab}(t,t_w). \tag{5.10}$$

The separation (5.7) induces within the MCA a similar separation for Σ and D [cf. Eqs. (5.3) and (5.4)], namely, for close times,

$$D_{ab}^{\mathrm{FDT}}(t-t_w) = \lim_{t_w\to\infty} D_{ab}(t,t_w),$$

$$\Sigma_{ab}^{\mathrm{FDT}}(t-t_w) = \lim_{t_w\to\infty} \Sigma_{ab}(t,t_w), \tag{5.11}$$

where FDT holds:

$$\Sigma_{ab}^{\mathrm{FDT}}(t-t_w) = \frac{1}{T}\frac{\partial D_{ab}^{\mathrm{FDT}}}{\partial t_w}(t-t_w). \tag{5.12}$$

Again, we can define

$$\lim_{t-t_w\to\infty}\ \lim_{t_w\to\infty} D^{\mathrm{FDT}}_{ab}(t-t_w)\equiv D^{\mathrm{EA}}_{ab}, \tag{5.13}$$

and mark with the tilde the aging part of the kernels:

$$D_{ab}(t,t_w)=\tilde{D}_{ab}(t,t_w), \quad \Sigma_{ab}(t,t_w)=\tilde{\Sigma}_{ab}(t,t_w). \tag{5.14}$$

In order to close the dynamical equations, we make an ansatz for the aging parts $\tilde{C}_{ab}(t,t_w)$ and $\tilde{R}_{ab}(t,t_w)$. For a problem in which the correlations vary only within two time scales, the natural generalization of the solution in [4] is

$$\tilde{C}_{ab}(t,t_w)=\tilde{\mathcal{C}}_{ab}[h_{ab}(t_w)/h_{ab}(t)], \tag{5.15a}$$

$$\tilde{R}_{ab}(t,t_w)=\frac{X_a}{T}\,\frac{\partial \tilde{C}_{ab}}{\partial t_w}\,(t,t_w), \tag{5.15b}$$

where the X_{ab} are constants and the $h_{ab}(t)$ are functions to be determined from the dynamical equations. Note that the derivative in Eq. (5.15) is taken with respect to the earliest time.

Remarkably, it turns out that one can close the equations with two different types of ansatz for the long-time aging behavior. In Appendix E we show how this is done. In terms of the effective temperatures the meaning of these two possibilities is the following.

(1) Thermalized aging regime. The effective temperatures associated with the observables O_1,O_2 are equal to each other for frequencies and waiting times in the aging regime: they are not necessarily equal to the temperature of the bath. At higher frequencies, they both coincide with the one of the bath:

$$\begin{aligned}
&T_1(\omega,t_w)=T_2(\omega,t_w)=T, \quad t_w\to\infty, \quad C_{ab}>C^{\mathrm{EA}}_{ab}, \quad \text{quasiequilibrium},\\
&T_1(\omega,t_w)=T_2(\omega,t_w)\neq T, \quad \omega\to 0, \quad t_w\to\infty, \quad C_{ab}<C^{\mathrm{EA}}_{ab}, \quad \text{aging}.
\end{aligned} \tag{5.16}$$

Not surprisingly, in this case we find that O_1 and O_2 are strongly coupled (*also in the aging regime*) in the sense that the mutual responses

$$\tilde{R}_{12}(t,t_w)=\frac{X}{T}\,\frac{\partial \tilde{C}_{12}}{\partial t_w}, \quad \tilde{R}_{21}(t,t_w)=\frac{X}{T}\,\frac{\partial \tilde{C}_{21}}{\partial t_w}, \tag{5.17}$$

where $X>0$, are of the same order of the self response functions.

(2) Unthermalized aging regime. The effective temperatures associated with the observables O_1,O_2 for combinations of frequencies and waiting times corresponding to the aging regime are neither equal to each other nor to that of the bath, while for higher frequencies they both coincide with the one of the bath.

$$\begin{aligned}
&T_1(\omega,t_w)=T_2(\omega,t_w)=T, \quad t_w\to\infty, \quad C_{ab}>C^{\mathrm{EA}}_{ab}, \quad \text{quasieq.},\\
&T_1(\omega,t_w)\neq T_2(\omega,t_w)\neq T, \quad \omega\to 0, \quad t_w\to\infty, \quad C_{ab}<C^{\mathrm{EA}}_{ab}, \quad \text{aging}.
\end{aligned} \tag{5.18}$$

In this case, O_1 and O_2 are effectively uncoupled (*in the aging regime*), in the sense that

$$\tilde{R}_{12}(t,t_w)=\frac{X_{12}}{T}\,\frac{\partial \tilde{C}_{12}}{\partial t_w}, \quad \tilde{R}_{21}(t,t_w)=\frac{X_{21}}{T}\,\frac{\partial \tilde{C}_{21}}{\partial t_w}, \tag{5.19}$$

where

$$X_{12},X_{21}\to 0. \tag{5.20}$$

We have not found any other way of closing the equations [31,32]. We shall show below, in a particular case which can be numerically solved, that indeed either case (1) or case (2) take place. Let us remark that a solution with $X_{12}=X_{21}\neq 0$, $X_{11}=X_{22}=X\neq 0$, and $X\neq X_{12}$ is not compatible with the interpretation of X as an inverse temperature. We do not find such a solution in our test models and believe that it is not realizable in general.

The considerations of this section can also be made in the limit of small stirring.

A. An explicit model with thermalization

In order to test in a particular example that these asymptotic solutions are not only consistent, but are in fact reached, we solve numerically the mode coupling equations with two coupled modes, with F given by

$$F(\mathbf{q})=q_{11}^{p}+K^2 q_{22}^{p}. \tag{5.21}$$

We impose normalization at equal times of the autocorrelation of both modes:

$$C_{11}(t,t)=C_{22}(t,t)=1. \tag{5.22}$$

As usual with the mode-coupling equations [(33)–(36) and (29)], one can find a disordered mean field model whose dynamics is exactly given by these equations. This is the model defined in Appendix A.

Figures 7 and 8 show the numerical solution of the exact system of coupled integrodifferential equations whose large-time asymptotic can be analytically obtained as in Appendix E. In Figs. 7 and 8 we plot $\chi(C)$ for two cases. In Fig. 7 we consider two uncoupled systems that evolve from the initial

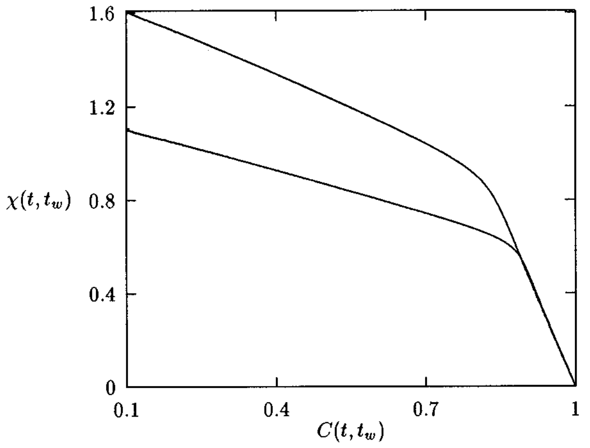

FIG. 7. The susceptibility $\chi_{11}(t,t_w)$ and $\chi_{22}(t,t_w)$ vs the corresponding autocorrelation functions $C_{11}(t,t_w)$ and $C_{22}(t,t_w)$ for the *uncoupled*, aging systems. The slopes of the curves, i.e., the FDT violation factors and hence the effective temperatures are different. This corresponds to the *unthermalized* case.

condition $C_{11}(0,0)=C_{22}(0,0)=1$ and $C_{12}(0,0)=C_{21}(0,0)=0$. We see that the systems evolve independently and $X_{11}\neq X_{22}$ while $X_{12}=X_{21}=0$ (unthermalized case). In Fig. 8 we consider the same global system but now including a weak coupling between the two individual systems. Clearly, after a short transient associated to short times, all curves $\chi_{ij}(C_{ij})$ get parallel. The systems aging to regime temperatures for the two subsystems have become equal (thermalized case).

It is interesting to note that, in this example, Onsager's reciprocity relations hold: $R_{12}=R_{21}$, $C_{12}=C_{21}$. However, one can imagine situations in which they do not hold separately, but where however $X_{12}=X_{21}$.

For comparison, we show in Fig. 9 the corresponding plot for two weakly coupled systems in the limit of small stirring.

B. Many scales

Let us briefly discuss what would happen in the presence of many scales, each one with its own temperature.

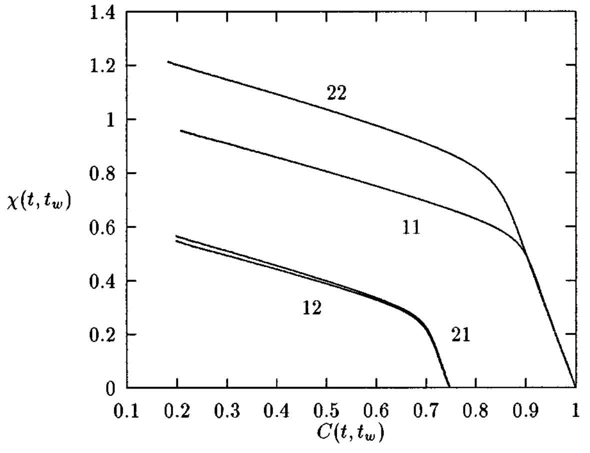

FIG. 8. The susceptibilities $\chi_{ij}(t,t_w)$ vs the correlation functions $C_{ij}(t,t_w)$ for the two aging systems of Fig. 2, this time *weakly* coupled. The FDT violation factors $X_{ij}(C_{ij})$ are almost parallel: thermalization is almost complete.

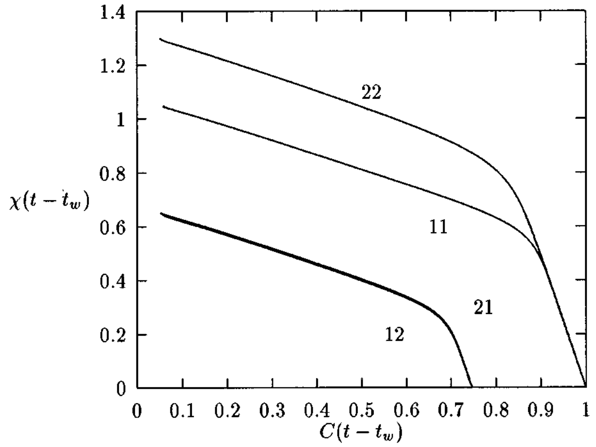

FIG. 9. The susceptibilities $\chi_{ij}(t-t_w)$ vs the correlation functions $C_{ij}(t-t_w)$ for two slightly driven systems ($D=0.1$), *weakly* coupled. The FDT violation factors $X_{ij}(C_{ij})$ given by the slopes of these curves are now the same: after a short transient corresponding to finite times all the curves become parallel. The aging regimes have *thermalized.*

Consider, for instance, two observables O_1 and O_2, their associated autocorrelation functions $C_{11}(t,t_w)$, $C_{22}(t,t_w)$, their integrated self-responses $\chi_{11}(t,t_w)$, $\chi_{22}(t,t_w)$ and the effective temperatures $T_1(C_{11})$, $T_2(C_{22})$. We can plot $T_1(C_{11})$ vs C_{11} and $T_2(C_{22})$ vs C_{22}. These two plots need not be the same, even if all scales are thermalized. Consider now a parametric plot of $C_{11}(t,t_w)$ vs $C_{22}(t,t_w)$, in the limit of very large t_w: it defines a function $C_{11}=\mathcal{H}(C_{22})$ that allows one to calculate C_{11} for large t,t_w, given $C_{22}(t,t_w)$.

The condition for thermalization in every time scale is then that the curve $T_2(C_{22})$ coincides with the curve $T_1(\mathcal{H}(C_{22}))$, both considered as functions of C_{22}:

$$T_1(\mathcal{H}(C_{22}))=T_2(C_{22}). \tag{5.23}$$

The deviation of $T_1(\mathcal{H}(C_{22}))$ from $T_2(C_{22})$ is a measure of the degree in which the two observables are not thermalized.

Actually, Eq. (5.23) was obtained as an ansatz for the dynamics of a manifold in a random medium within the Hartree approximation [37], where the role of different observables is played by the displacements at different spatial wavelengths:

$$C_k(t,t')=\mathcal{H}_k(C_{k=0}),$$

$$T_k(\mathcal{H}_k(C_{k=0}))=T_{k=0}(C_{k=0}). \tag{5.24}$$

In this case, the modes at different k thermalize at the same effective temperature, although their mutual responses vanish, as implied by translational symmetry in mean.

VI. COMPARISON WITH OTHER EFFECTIVE TEMPERATURES

The idea of ''fictive temperatures'' [38] T_f in glasses goes back to the 1940s and it has developed since [39–41]. Here we recall it briefly, for the sake of comparison with the effective temperature that we have discussed.

When cooling a liquid the time needed to establish equilibrium grows and, eventually, the structural change cannot keep pace with the rate of cooling: the system falls out of equilibrium and enters the glass transition region. It is then said that ''the structure is frozen'' at a temperature characterized by a fictive temperature T_f. The fictive temperature defined in terms of different quantities of interest, e.g., the enthalpy, the thermal expansion coefficient, etc., do not necessarily coincide. Furthermore, it has been experimentally observed that glasses with the same fictive temperatures arrived at through different preparation paths may have different molecular structures. The fictive temperature is hence a phenomenological convenience and should not be associated with a definite molecular structure [41].

The fictive temperature is a function of the temperature of the bath T. At high temperature, when the sample is in the liquid phase, $T_f = T$. When the liquid enters the transition range T_f departs from T and $T_f > T$; and, finally, deep below the transition range, where the relaxation is fully stopped, $T_f \to T_g$. The detailed bath-temperature dependence of the fictive temperature in the region of interest is usually expressed by [40,41]

$$\dot{T}_f = -\frac{T_f - T}{\tau(T_f, -T)}, \tag{6.1}$$

where the characteristic time $\tau(T_f, T)$ depends both upon $T_f(t)$ and upon the thermal history of the sample given by $T(t)$. At equilibrium $T_f = T$ and $\dot{T}_f = 0$. This nonlinear differential equation determines $T_f(t)$ once one *chooses* $\tau(T_f, T)$. A commonly used expression is the Narayanaswamy-Moynihan equation

$$\tau(T_f, T) = \tau_0 \exp\left[\frac{xA}{T} + \frac{(1-x)A}{T_f}\right], \tag{6.2}$$

where τ_0, A and $x \in [0,1]$ are some constants. All the information about the dynamics of the system enters into T_f through these constants.

In order to obtain the time relaxation of the quantity of interest, the picture is completed by proposing a given relaxation function, like the stretched exponential, the Davidson-Cole function, etc., and by introducing the fictive temperature through the characteristic time $\tau(T_f, T)$. See [41] for an extensive discussion about the applications of T_f to the description of experimental data.

One may wonder whether is a relation between this *fictive* temperature of glass phenomenology and the *effective* temperature we have been discussing in this paper.

First of all, one notes that the fictive temperatures are defined through the relaxation of the observables, unlike the one we consider here which are defined rather in terms of fluctuations and responses. Both temperatures may depend upon the observable and upon the thermal history of the sample.

The dependence of the effective temperature defined in Eq. (3) upon T depends on the model. In all cases FDT holds in the high-temperature phase and $T_O(\omega, t_w) = T$, the effective temperature is equal to the bath temperature. When entering the low-temperature phase, the temperature dependence of the effective temperature observed at fixed low frequency depends on the model. In certain simple mean-field (or low-temperature mode-coupling) models whose dynamics following a quench into the glass phase can be solved, one can compute $T_O(\omega, t_w)$ explicitly. Three different behaviors are found.

(1) In the simple model we have been using as a test example in the previous sections [4], $T_O(\omega, t_w)$ starts from $T_O(\omega, t_w) = T_g$ at $T = T_g$ and slightly *increases* when the bath temperature decreases below T_g. One would instead expect a fictive temperature to remain stuck to T_g in a (mean-field) model in which the glass transition is sharp.

(2) There are other mean-field models such as the Sherrington-Kirkpatrick spin-glass model in which there are infinitely many different effective temperatures. The lowest aging-regime temperature appears discontinuously $[T_O(\omega, t_w) > T_g]$ as one crosses T_g and can be shown [5] to *decrease* with decreasing temperature. Another example of this kind is the model of a particle moving in a random potential with long-range correlations [20,25].

(3) In all cases we know of domain growth models [19] one obtains $X(C) \to 0$ for $t_w \to \infty$, in the whole low-temperature phase. Hence $T_O(\omega, t_w) \to \infty$ in the aging (coarsening) regime for all heat bath temperatures below the ordering transition. This behavior also holds for certain extremely simple disordered systems such as the spherical Sherrington-Kirkpatrick spin glass [42], the toy domain growth model we used in this paper.

It is important to remark that these simple examples have the (sometimes unrealistic) feature that nothing depends permanently upon the cooling procedure. One expects, however, that more refined models that go beyond the mean-field approximation will capture a cooling rate dependence that will also become manifest in the effective temperature.

Another attempt to identify and relate a microscopic effective temperature to the fictive temperature of glasses was put forward by Baschnagel, Binder, and Wittmann [43] in the context of a lattice model for polymer melts. They have pointed out that in this model the usual FDT relation between the specific heat with the energy fluctuations is broken at low temperatures and have tried to identify the FDT violation factor with an expression they propose for an *internal temperature*. The internal temperature defined in [43] has, however, the unpleasant property of not reducing to the bath temperature in the high temperature phase.

In [44] Franz and Ritort have also discussed the possibility of relating the FDT ratio with an effective temperature, in the particular case of the Backgammonn model. They have compared the value of $X(t, t_w)$ for finite times t, t_w with the temperature arising from an adiabatic approximation they used to solve the model. The result is negative, in the sense that $T/X(t, t_w)$ does not coincide with the ''adiabatic temperature.''

These examples suggest that the phenomenological fictive temperatures act essentially as parameters for describing an out-of-equilibrium ''equation of state'' while the effective temperature we have discussed plays a role closer to the thermodynamical one.

VII. DISCUSSION, EXPERIMENTAL PERSPECTIVES, AND CONCLUSIONS

Although our discussion has been biased by the models we can solve at present, we feel that the concept of the ef-

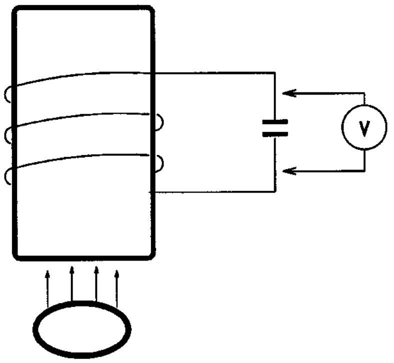

FIG. 10. An effective temperature measurement for a magnetic system. The coil is wound around the sample, which is in contact with the bath. The coil and capacitor have zero resistance.

fective temperature that we discussed should be relevant for many systems *with small energy flows*. Indeed, the key observation we make is that this temperature actually controls thermalization and heat flows within a time scale. Therefore, the effective temperature can be a starting point for the investigation of the thermology and, hopefully, the thermodynamics of out-of-equilibrium systems with small energy flows.

In order to use this idea as a guiding concept for the planning of experiments, one has to take some care: obviously, waiting times can be large but not infinite in experiments, and stirring rates can be small but not infinitesimal. Moreover, the models that we have explicitly discussed do not exhibit dependence on the cooling history of the sample. Now, one would expect in general the effective temperature to depend on t,t_w and, e.g., on the cooling rate. One can hope to catch this aspect in more refined models.

One should also pay attention to the equilibration times. The effective temperature measured by a small oscillator is related to the FDT violation factor provided that the equilibration time t_c [Eq. (3.10)] remains much smaller than t_w. It sometimes happens that an observable O which should thermalize at an effective temperature $T_O(\omega,t_w)$ exhibits so small fluctuations at this frequency that this time becomes unbearably long. This is the case, for instance, of high k modes in the aging time scale for a manifold in a random potential. Therefore, although all k modes nominally thermalize at the same effective temperature, only low enough k modes can be effectively used to measure it. How low k must be will depend on the time scale one is looking at.

Several numerical and real experiments in structural glasses can be envisaged. For example, one can compare density fluctuations and compressibility in different length scales in order to check if they are equilibrated within a time scale. Since the low-temperature extension of the MCT makes definite predictions on the value of the first nontrivial effective temperature appearing as one crosses the transition (see Sec. VI), this provides a concrete ground for an experimental testing of MCT.

We close by considering the following (slightly *Gedanken*) spin-glass experiment of Fig. 10: Currents are induced in the coil by the magnetization noise of the spin glass, which is in contact with a heat bath. Apart from the interaction with the sample, the L-C circuit of coil and capacitor is without losses. This is exactly a realization of the oscillator as a thermometer of Sec. II. From what we know from the time scales of real spin glasses [16,18], if the time after the quench is of the order of 10 min, and the period of the L-C circuit is of the order of the second, we are probing (at least partially) the aging regime: the temperature (defined as the average energy of the capacitor) should be different from the bath temperature. We believe that it would be interesting to return to the magnetization noise experiments [17] with the purpose of measuring the effective temperature: this would give us, for instance, useful insights into the nature of the spin-glass transition.

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APPENDIX A: THE TEST MODEL

Throughout this paper we have used as a test model a spherical disordered model with p-spin interactions. The model has been introduced in its purely relaxational version by Crisanti and Sommers [45] as a simple spin-glass model with several advantages, in particular, that exact dynamical equations can be written for it in the thermodynamic limit. As shown by Franz and Hertz [36], these dynamical equations are also those obtained from the MCA to the Amit-Roginsky model [34]. When considered in full generality, the two-time dynamical equations correspond to the low temperature extension [36,29] of the simplest mode coupling theory for the supercooled liquid phase proposed by Leutheusser [46] and Bengtzelius, Gotze, and Sjölander [47,30]. A thorough discussion of the physical principles underlying mode-coupling theory is found in [48], Chap. 9. It has been also recently shown by Chandra *et al.* that this model is related to a mean-field approach to Josephson junction arrays [49].

We consider a system of N variables $\mathbf{s}=(s_1\ldots,s_N)$, subject to forces F_i^J given by

$$F_i^J(\mathbf{s})=\sum_{\{j_1,\ldots,j_{p-1}\}} J_i^{j_1\cdots j_{p-1}} s_{j_1}\cdots s_{j_{p-1}}, \tag{A1}$$

where the couplings are random Gaussian variables. For different sets of indices $\{i,j_1,\ldots,j_{p-1}\}$ the J's are uncorrelated, while for permutations of the same set of indices they are correlated so that

$$\overline{F_i^J(\mathbf{s}')F_j^J(\mathbf{s})}=\delta_{ij}f_1(q)+s_i s_j' f_2(q)/N, \tag{A2}$$

where $q=(\mathbf{s}\cdot\mathbf{s}')/N$. In the purely relaxational case, one has $f_2(q)=f_1'(q)$. We take here $f_2(q)=(1-D)f_1'(q)$, where

$f_1(q)=pq^{p-1}/2$. The couplings $J_i^{j_1\cdots j_{p-1}}$ are symmetric under the permutation $i\leftrightarrow j_k$ in the purely relaxational case ($D=0$). On the other hand, if $J_i^{jj_2\cdots j_{p-1}}$ and $J_j^{ij_2\cdots j_{p-1}}$ are uncorrelated, one has $D=1$.

The dynamics is of the Langevin type:

$$\dot{s}_i=-F_i^J(\mathbf{s})-\mu(t)s_i+h_i(t)+\eta_i(t), \tag{A3}$$

where η is a white noise of variance $2T$, $\mu(t)$ is a Lagrange multiplier enforcing the spherical constraint $\Sigma_{i=1}^N s_i^2=1$, and $h_i(t)$ is an external field (usually set to zero), needed to define the response functions.

In Figs. 1 and 2 we plot the $\chi(C)$ curves for the asymmetrical and symmetrical $p=3$ model, respectively. In Figs. 3 and 4 we plot the autocorrelation decays for the symmetrical and asymmetrical versions, respectively. Figures 5 and 6 show the $\chi(C)$ curves for the $p=2$ version that is analogous to the $O(N)$ model in $D=3$ when $N\to\infty$.

In order to check thermalization in this particular example we consider the evolution of two such systems with spins s and σ, and uncorrelated realizations of disorder J and J' and thermal noise η and η', respectively. They are coupled via the term proportional to μ_{12} and μ_{21} in the Langevin equations

$$\frac{\partial s_i}{\partial t}=-F_i^J(s)-\mu_{11}(t)s_i-\mu_{12}\sigma_i+h_i+\eta_i\,, \tag{A4a}$$

$$\frac{\partial \sigma_i}{\partial t}=-KF_i^{J'}(\sigma)-\mu_{22}(t)\sigma_i-\mu_{21}s_i+h_i'+\eta_i'\,. \tag{A4b}$$

We set $\mu_{12}=\mu_{21}$, so that the coupling does not contribute to the stirring. The coefficients μ_{ii} are the Lagrange multipliers for each system. The factor K is introduced to break the symmetry between the subsystems. The correlations

$$C_{11}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \langle s_i(t)s_i(t_w)\rangle,$$

$$C_{22}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \langle \sigma_i(t)\sigma_i(t_w)\rangle,$$

$$C_{12}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \langle s_i(t)\sigma_i(t_w)\rangle,$$

$$C_{21}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \langle s_i(t)\sigma_i(t_w)\rangle \tag{A5}$$

and responses

$$R_{11}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \delta\langle s_i(t)\rangle/\delta h_i^s(t_w),$$

$$R_{22}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \delta\langle \sigma_i(t)\rangle/\delta h_i^\sigma(t_w),$$

$$R_{12}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \delta\langle s_i(t)\rangle/\delta h_i^\sigma(t_w),$$

$$R_{21}(t,t_w)=\frac{1}{N}\sum_{i=1}^N \delta\langle s_i(t)\rangle/\delta h_i^s(t_w) \tag{A6}$$

precisely satisfy Eqs. (5.1a)–(5.4) with F given by Eq. (5.21).

The results for the effective temperatures obtained from the numerical integration of the exact evolution equations of these systems are shown in Figs. 7 and 8. See the main text for the discussion.

APPENDIX B: ENERGY OF THE OSCILLATOR

In this appendix we solve Eq. (3.8), we compute $\frac{1}{2}\omega_0^2\langle x^2(t)\rangle$, the average potential energy of the oscillator. We thus prove Eq. (3.14) and its form Eq. (3.13) valid for the stationary case.

Let us define

$$\chi(\omega,t)\exp(i\omega t)\equiv\int_0^t dt'R(t,t')\exp(i\omega t'), \tag{B1a}$$

$$C(\omega,t)\exp(i\omega t)\equiv\int_0^t dt'C(t,t')\exp(i\omega t'). \tag{B1b}$$

If $\omega^{-1}\ll t$, we can assume that $\chi(\omega,t)$ and $C(\omega,t)$ are functions that vary slowly with t, thus defining a Fourier component that is ''local'' in time t.

In general

$$\langle x^2(t)\rangle=a^2\int_0^t dt'\int_0^t dt''G(t,t')G(t,t'')C(t',t''), \tag{B2}$$

where $G(t,t')$ is the Green's function for the oscillator plus the term representing the response of the system and $G(t,t')$ is the system's auto-correlation function. Using the definition in Eq. (B1a) one can show that the damped oscillator's Green's function reads

$$G(\omega,t)=\frac{1}{-\omega^2+\omega_0^2-a^2\chi''(\omega,t)}, \tag{B3}$$

$$G(t,t'')=\exp\left(-\frac{t-t'}{t_c(t)}\right)\sin[\omega_0(t-t')]\theta(t-t'). \tag{B4}$$

We have here replaced $\chi(t,\omega)$ by $\chi''(t,\omega)$ using the fact that $a^2N\ll1$. The characteristic time $t_c(t)$ of the damped oscillator is given by

$$t_c(t)=\frac{2\omega_0}{a^2\chi''(\omega_0 t)}. \tag{B5}$$

We can now study Eq. (B2) by using the above expressions for $G(t,t')$. After a simple change of variables, using causality and the fact that $G(t,t')$ decays exponentially as a function of time differences, Eq. (B2) can be rewritten as

$$\langle x^2(t)\rangle = a^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' G(t,t-\tau) G(t,t-\tau') C(\omega,t-\tau) \exp[i\omega(\tau-\tau')]. \tag{B6}$$

The fast exponential decay of the Green's function allows us to replace $C(\omega,t-\tau)$ by $C(\omega,t)$. Thus,

$$\begin{aligned}\langle x^2(t)\rangle &= a^2 \int_{\infty}^{\infty} \frac{d\omega}{2\pi} G(\omega,t) G(-\omega,t) C(\omega,t)\\ &= a^2 \int_{\infty}^{\infty} \frac{d\omega}{2\pi} \frac{C(\omega,t)}{\chi''(\omega,t)-\chi''(-\omega,t)} \left[\frac{1}{\omega_0^2-\omega^2-a^2N\chi''(-\omega,t)} - \frac{1}{\omega_0^2-\omega^2-a^2N\chi''(\omega,t)}\right].\end{aligned} \tag{B7}$$

This integral can be calculated by the method of residues. We can close the circuit on the upper complex half plane. Since $\chi''(\omega,t)$ is analytic in the upper half plane, the only singularities are the zeroes of the denominators. Assuming that $a^2\chi''$ is small, they lie in the vicinity of $\omega=\pm\omega_0$. In fact one can check that only two poles penetrate inside the circuit of integration. We obtain therefore

$$\langle x^2(t)\rangle = \frac{2\tilde{C}(\omega_0,t)}{\omega_0 \chi''(\omega_0,t)}. \tag{B8}$$

This is the general result for the temperature. The particular result (3.13) that holds for the stationary case is recovered from Eq. (B8) by letting $\tilde{C}(t,\omega)$ and $\chi''(t,\omega)$ be independent of t. Thus, Eq. (B8) reduces to Eq. (3.13).

APPENDIX C: SMALL BUT MACROSCOPIC THERMOMETERS

In this appendix we show that the thermometric considerations made in Sec. III do not crucially depend on the choice of an oscillator as a thermometer. We use here a small but *macroscopic* thermometer defined by the variables y_i, $i=1,\ldots,n$, and we couple it only to the observable $O(s)$ of the system through a degree of freedom $x(y)$.

Our measurement procedure is as follows: we first thermalize the thermometer with an auxiliary bath at temperature T^*. We then disconnect it from the bath and we connect it to the system through O. If there is no flow of energy between thermometer and system, then we conclude that the measured temperature is T^*.

The energy of the thermometer plus its coupling with the system is

$$H = E(y) - aO(s)x(y). \tag{C1}$$

The net power gain of the thermometer is then $\dot{Q}(t)$, given by

$$\dot{Q} = a\langle \dot{O}x\rangle = a\partial_{t'}\langle x(t)O(t')\rangle|_{t'\to t^-}. \tag{C2}$$

We look for the condition that ensures stationarity for the thermometer. The thermometer is characterized by a temperature-dependent correlation $C_x(t,t')=\langle x(t)x(t')\rangle$ and its associated response $R_x(t,t')$. Using linear response one has

$$O(t) = O_b(t) + a\int_0^t dt' R_O(t,t')x(t'), \tag{C3a}$$

$$x(t) = x_b(t) + a\int_0^t dt' R_x(t,t')O(t'). \tag{C3b}$$

To leading order in a, $\langle Ox\rangle$ is given by

$$\langle Ox\rangle = a\int_0^t dt''[R_x(t,t'')C_O(t'',t') + R_O(t',t'')C_x(t,t'')]. \tag{C4}$$

Assuming now that T^* is such that the thermometer can be considered to be almost in equilibrium (note that the coupling a is small and that we have chosen a thermometer that is not itself a glass), we obtain

$$\dot{Q} = a^2\int_0^t dt' R_x(t-t')\left(\frac{\partial C_O(t,t')}{\partial t'} - T^* R_O(t,t')\right). \tag{C5}$$

The condition for having no flow is then that the average of the parenthesis in the integral is zero. The weight function for this average is $R_x(t-t')$ which contains the characteristic time of the thermometer.

APPENDIX D: DEFINITION OF TIME CORRELATION SCALES

In this appendix we review briefly the definition of correlation "time scale" introduced in [5] for a correlation function that depends nontrivially upon two times.

Given a correlation function $C(t,t_w)$, which we assume normalizable in the large-time limit $C(t,t)\to C_\infty>0$, we consider three increasing times $t_1<t_2<t_3$, and the limit in which they all go to infinity, but in a way to keep $C(t_3,t_2)=a$ and $C(t_2,t_1)=b$ const. We thus define the limit

$$\lim_{\substack{t_1,t_2,t_3\to\infty\\ C(t_3,t_2)=a,\,C(t_2,t_1)=b}} C(t_3,t_1) \equiv f(a,b). \tag{D1}$$

The mere existence of the limit "triangle relation" f has extremely strong consequences: considering four times one can easily show that f is associative

$$f(a,f(b,c)) = f(f(a,b),c). \tag{D2}$$

The form of an associative function on the reals is very restricted, and a classification of all possible forms can be made [5].

It is sometimes convenient to work with the ''inverse'' $\overline{f}$ of f defined as

$$f(a,b)=c \Rightarrow \overline{f(a,c)}=b. \tag{D3}$$

We can now define a correlation scale in the following way: given two values of the correlation at large times $C(t,t_w)=\mathcal{C}^1$ and $C(t,t'_w)=\mathcal{C}^2$, $t>t'_w>t_w$ and $\mathcal{C}^1<\mathcal{C}^2$, they are in a different correlation scale if

$$\overline{f(\mathcal{C}^2},\mathcal{C}^1)=\mathcal{C}^1, \tag{D4}$$

and are in the same scale otherwise. In other words, the time it takes the system to achieve $\mathcal{C}^2$ is negligible with respect to the time it takes to achieve $\mathcal{C}^1$.

APPENDIX E: SOLUTION OF THE TWO-COUPLED-MODE EQUATION

Using the separation (5.7)–(5.14) and the mode-coupling approximation (5.3) and (5.4) we obtain a similar separation for D_{ab} and Σ_{ab}:

$$D_{ab}^{\mathrm{FDT}}(t-t_w)=F_{ab}(\mathsf{C}^{\mathrm{FDT}}(t-t_w)), \tag{E1a}$$

$$\Sigma_{ab}^{\mathrm{FDT}}(t-t_w)=\sum_{c,d} F_{ab,cd}(\mathsf{C}^{\mathrm{FDT}}(t-t_w))R_{cd}^{\mathrm{FDT}}(t-t_w), \tag{E1b}$$

where $\mathsf{C}^{\mathrm{FDT}}$ stands for the set $C_{ab}^{\mathrm{FDT}}(t-t_w)$. One similarly obtains, in the aging regime,

$$\widetilde{D}_{ab}(t,t_w)=F_{ab}(\widetilde{\mathsf{C}}(t,t_w)), \tag{E2a}$$

$$\widetilde{\Sigma}_{ab}(t,t_w)=\sum_{c,d} F_{ab,cd}(\widetilde{\mathsf{C}}(t,t_w))\widetilde{R}_{cd}(t,t_w). \tag{E2b}$$

We can now write two coupled sets of equations, valid in the quasiequilibrium regime and in the aging regime, respectively. For $t-t_w$ finite, and large times $t>t_w$, we have

$$\frac{\partial C_{ab}^{\mathrm{FDT}}(t-t_w)}{\partial t}=-\sum_c \left[\mu_{ac}^{\infty}+\frac{D_{ac}^{\mathrm{FDT}}(0)}{T}\right]C_{cb}^{\mathrm{FDT}}(t-t_w)-\frac{1}{T}\sum_c D_{ac}^{\mathrm{EA}}C_{cb}^{\mathrm{EA}}+M_{ab}^{\infty}+\frac{1}{T}\sum_c \int_{t_w}^{t} dt'' D_{ac}^{\mathrm{FDT}}(t-t'')\,\frac{\partial C_{cb}^{\mathrm{FDT}}(t''-t_w)}{\partial t_w}, \tag{E3}$$

where $\mu_{ac}^{\infty}\equiv\lim_{t\to\infty}\mu_{ac}(t)$ and

$$M_{ab}^{\infty}\equiv\sum_c \lim_{t\to\infty}\int_0^t dt''[\widetilde{D}_{ac}(t,t'')\widetilde{R}_{cb}(t,t'')+\widetilde{\Sigma}_{ac}(t,t'')\widetilde{C}_{cb}(t,t'')]. \tag{E4}$$

In the aging regime, for $t>t_w$ we have

$$\begin{aligned}\frac{\partial \widetilde{C}_{ab}(t,t_w)}{\partial t}=&-\sum_c \left[\mu_{ac}(t)+\frac{D_{ac}^{\mathrm{FDT}}(0)-D_{ac}^{\mathrm{EA}}}{T}\right]\widetilde{C}_{cb}(t,t_w)+\sum_c \widetilde{D}_{ac}(t,t_w)\,\frac{C_{cb}^{\mathrm{FDT}}(0)-C_{cb}^{\mathrm{EA}}}{T}+\sum_c \int_0^{t_w} dt''\widetilde{D}_{ac}(t,t'')\widetilde{R}_{cb}(t_w,t'')\\&+\sum_c \int_0^t dt''\widetilde{\Sigma}_{ac}(t,t'')\widetilde{C}_{cb}(t'',t_w),\end{aligned} \tag{E5a}$$

$$\frac{\partial \widetilde{R}_{ab}(t,t_w)}{\partial t}=-\sum_c \left[\mu_{ac}(t)+\frac{D_{ac}^{\mathrm{FDT}}(0)-D_{ac}^{\mathrm{EA}}}{T}\right]\widetilde{R}_{cb}(t,t_w)+\sum_c \int_{t_w}^{t} dt''\widetilde{\Sigma}_{ac}(t,t'')\widetilde{R}_{cb}(t'',t_w)+\sum_c \widetilde{\Sigma}_{ac}(t,t_w)\,\frac{C_{cb}^{\mathrm{FDT}}(0)-C_{cb}^{\mathrm{EA}}}{T}. \tag{E5b}$$

Equation (E4), for given M_{ab}^{∞}, is very similar to the high-temperature mode-coupling equations [30], and can be solved in the same way. An asymptotic solution for the aging regime can be obtained by using the generalization to more than one mode of the ansatz in [4], Eq. (5.15). The derivative terms in Eqs. (E5) can be then dropped provided that $X_{11}\neq 0$ and $X_{22}\neq 0$, a fact to be verified *a posteriori*. We shall find in this way two different solutions for Eqs. (E5).

In the unthermalized case X_{11} and X_{22} are different from zero, and possibly different from each other, while $X_{12}=X_{21}=0$. It is then easy to see that Eqs. (E5) become effectively uncoupled *in this regime*, and can be solved [4] as two separated one-mode equations, with the ansatz

$$\widetilde{C}_{aa}(t,t_w)=\widetilde{\mathcal{C}}_{aa}(h_{aa}(t_w)/h_{aa}(t)), \tag{E6a}$$

$$\widetilde{R}_{aa}(t,t_w)=\frac{X_{aa}}{T}\,\frac{\partial \widetilde{C}_{aa}}{\partial t_w}\,(t,t_w), \tag{E6b}$$

$$\widetilde{C}_{12}(t,t_w)=\widetilde{C}_{21}(t,t_w)=0. \tag{E6c}$$

In the thermalized case, one assumes

$$X_{11}=X_{22}=X_{12}=X_{21}=X\neq 0, \tag{E7a}$$

$$h_{11}=h_{22}=h_{12}=h_{21}=h. \tag{E7b}$$

Making the change of variables

$$\lambda\equiv\frac{h(t_w)}{h(t)}, \quad \lambda'\equiv\frac{h(t'')}{h(t)}, \tag{E8}$$

one finds that only the dependence on λ survives in the equations, and that they reduce to a set of four (instead of eight) consistent equations for the correlations (see [4,25] for a systematic approach).

In this way, both for the thermalized and the unthermalized case one can obtain X_{ab}, M_{ab}, and G_{ab}^{EA}. One then has to check that $X_{11}\neq 0$ and $X_{22}\neq 0$. If this is not the case the equations become identities, and one cannot anymore neglect the derivative term. One has therefore to use a more refined long-time limit. These values have to be substituted in Eq. (E4), in order to complete the solution in both regimes.

Let us remark here that the problem of selecting the functions h_{ab} remains open. This is an asymptotic matching problem in a non-local equation, and does not appear to be easily solvable.

We have thus found the long-time limit of the correlations and responses. If there is more than one asymptotic solution (even modulo h), we do not know for the time being which asymptotic form is selected by the unique solution of the evolution equations, without resorting to explicit numerical integration.

In short we have the following.

(1) Unthermalized aging regime:

$$X_{12}\to 0, \quad X_{21}\to 0, \quad X_{11}\neq X_{22}. \tag{E9}$$

We have, therefore,

$$\tilde{R}_{11}(t,t_w)=\frac{X_{11}}{T}\frac{\partial\tilde{C}_{11}}{\partial t_w}, \quad \tilde{R}_{22}(t,t_w)=\frac{X_{22}}{T}\frac{\partial\tilde{C}_{22}}{\partial t_w}, \tag{E10}$$

and

$$\tilde{\Sigma}_{11}(t,t_w)=\frac{X_{11}}{T}\frac{\partial\tilde{D}_{11}}{\partial t_w}, \quad \tilde{\Sigma}_{22}(t,t_w)=\frac{X_{22}}{T}\frac{\partial\tilde{D}_{22}}{\partial t_w}. \tag{E11}$$

(2) Thermalized aging regime:

$$X_{11}=X_{12}=X_{21}=X_{22}\equiv X, \tag{E12}$$

$$\tilde{\Sigma}_{ab}(t,t_w)=\frac{X}{T}\frac{\partial\tilde{D}_{ab}}{\partial t_w}, \quad \forall a,b, \tag{E13}$$

where the aging time scales ‘‘lock in,’’ i.e., there is the *same* function $h(t)$ for all a, b, such that

$$\tilde{C}_{ab}(t,t_w)=\tilde{C}_{ab}(h(t_w)/h(t)). \tag{E14}$$

This property can also be stated by saying that as $t,t_w\to\infty$ a plot of $\tilde{C}_{11}(t,t_w)$ vs $\tilde{C}_{22}(t,t_w)$ yields a single smooth curve: i.e., that there is a function $\mathcal{H}(C)$ such that

$$\tilde{C}_{11}(t,t_w)=\mathcal{H}(\tilde{C}_{22})(t,t_w). \tag{E15}$$

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